Free semimodules and their examples

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Abstract

The following thesis will cover *free semimodules* and their examples. We will define various algebraic structures via Ω-algebras (from universal algebra) with an emphasis on semimodules. We continue to define, categorically, the property of an algebraic structure being *free*, and show that this is a *universal property.* We construct the free semimodule on a set and explore various examples of them appearing in mathematics. We observe that the free semimodule on a set is a universal arrow from that set to the forgetful functor and that there is a functor called the *free functor*, that is the left adjoint to the forgetful functor, which provides an equivalent characterisation of the property of being free, that is when the object is in the image of the free functor.

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1 Introduction

There are seemingly disconnected properties of mathematical structures that can be unified under a general definition that characterises that property categorically, that is in each case. Category theory provides us with tools to form these more generalised characterisations.

One such characterisation of a property is that of an algebra being *free*. We can define this property, categorically, by using Ω-algebras to define algebraic categories and then for any algebraic category $\mathbb{C}$, we form the category $\mathbb{C}[X]$ of tuples consisting of a $\mathbb{C}$-algebra[1] and a map from the set X to the underlying set of the $\mathbb{C}$-algebra. The initial object in this category is the *free* $\mathbb{C}$*-algebra on* X. Throughout, we will return to vector spaces, the motivating example for this property since every vector space turns out to be free and this is equivalent to the proposition that every vector space has a basis.

We begin by defining Ω-algebras, a generalised definition of certain algebras that comes from *universal algebra*, and providing examples that will be used later. After this we are able to define a *free algebra* and the *free semimodule* on a set X. Semimodules are a generalisation of vector spaces where we relax the requirement of the vector space being defined over a field and instead define it over an arbitrary semiring. Once we define the free semimodule on a set X, we proceed with examples showing that every vector space has a basis and providing a categorical formulation of the *fundamental theorem of arithmetic.*

It is observed that the free semimodule on a set X is a *universal arrow* from X to the forgetful functor $\mathbf{SMod}_R \to \mathbf{Sets}$ and the property of an algebraic structure being free means, that algebra is in the image of the left adjoint to the forgetful functor, called the *free functor.*

2 Ω-algebras

Let Ω be a graded set (a set Ω equipped with a map $l_\Omega : \Omega \to \mathbb{N}$), $\Omega_0 = l_\Omega^{-1}(0), \ldots, \Omega_n = l_\Omega^{-1}(n), \ldots,$ and E a set of identities. The graded set Ω is referred to as the *signature*[2] and the map l_Ω assigns to each ω in Ω an *arity* n. An (Ω, E)-algebra, also referred to as a *universal algebra* or an *algebraic*

[1] Objects of an algebraic category $\mathbb{C}$ are referred to as $\mathbb{C}$-algebras.

[2] We can think of Ω as a sequence of disjoint sets $(\Omega_n = \{\omega \in \Omega : l_\Omega(\omega) = n\})_{n\in\mathbb{N}}$ in order to avoid using the symbol Ω [Jan20].

structure, is a pair (A, v) where A is a set and v is an action of Ω on A [ML71]. The action v, referred to as the *algebraic structure* on A, is a function which assigns to each operation of arity n, an n-ary operation $v_\omega : A^n \to A$. The map v can also be considered as a family $(v_\omega : A^{l(\omega)} \to A)_{\omega\in\Omega}$ of operations on A, or equivalently a map $v : \Omega \to \bigcup_{n\in\mathbb{N}} A^{A^n}$. We will refer to (Ω, E)-algebras as Ω-algebras and specify the identities we impose on the operators.

The convention is to use $\omega(a_1, \ldots, a_n)$ to refer to $v_\omega(a_1, \ldots, a_n)$ and $A = (A, v)$. This allows us to remove Ω and v from the notation. For example, a monoid M is an Ω-algebra with

$$\Omega_n = \begin{cases} \{e\}, & \text{for } n = 0, \\ \{m\}, & \text{for } n = 2, \\ \emptyset & \text{for } 0 \neq n \neq 2 \end{cases}$$

written $M = (M, e, m)$ rather than (M, v), where e and m are not elements of Ω but refer to their images under v [Jan15]. We will use this convention in our definitions of various algebraic structures.

From Ω we form the set Λ of all *derived operators*. As Mac Lane writes, for an operator ω of arity n, and n derived operators $\lambda_1, \ldots, \lambda_n$ of arities $m_1, \ldots, m_n$, the composite $\omega(\lambda_1, \ldots, \lambda_n)$ is a derived operator of arity $m_1 + \cdots + m_n$ [ML71]. Briefly, as an example, consider the property of associativity of addition, in order to formalise this property as an identity, we use derived operators to form a ternary operator, say λ, defined by $v_\lambda(a, b, c) = a + (b + c)$ for every a, b, and c in A. We form a second ternary operator μ defined as $v_\mu(a, b, c) = (a + b) + c$. Associativity can then be imposed on the Ω-algebra by requiring that $v_\lambda = v_\mu$. This motivates our inclusion of the set E of identities in our definition of an Ω-algebra.

The set E of *identities* for algebraic structures is a set of ordered pairs (λ, μ) of derived operators, where the derived operators of each pair have the same arity. An action v of Ω on A satisfies the identity (λ, μ) if $v_\lambda = v_\mu : A^n \to A$ [ML71]. An (Ω, E)-algebra is a set A together with an action v of Ω on A which satisfies all the identities of E [ML71].

An Ω-*algebra homomorphism* from A to B is a map $f : U(A) \to U(B)$ of underlying sets (where U is the forgetful functor $\mathbf{Alg}(\Omega) \to \mathbf{Sets}$), such that

$$f(\omega(a_1, \ldots, a_n)) = \omega(f(a_1, \ldots, a_n))$$

for every natural number n, every ω in Ω_n, and every $a_1, \ldots, a_n$ in A [Jan20]. The class of all Ω-algebras[3], with Ω-algebra homomorphisms as morphisms

[3] More accurately, the collection of all *small* Ω-algebras forms a category [ML71].

and composition defined as the usual composition of homomorphisms, forms a category called the category of Ω-algebras and is denoted by $\mathbf{Alg}(\Omega)$ [Jan20]. Categories of this form, for various Ω are called *algebraic categories*. A morphism is an *isomorphism* in $\mathbf{Alg}(\Omega)$ if and only if the map f is bijective on the underlying sets [Jan15]. We are able to transport an algebra structure along a bijection making it an isomorphism in $\mathbf{Alg}(\Omega)$. That is, if $A = (A, v)$ is an Ω-algebra, B a set, and $f : U(A) \to B$ a bijection, then there exists a unique Ω-algebra structure w on B making f an Ω-algebra isomorphism $(A, v) \to (B, w)$ [Jan15].

3 Examples of Ω-algebras

The following are examples of Ω-algebras that will be used throughout this thesis.

A *monoid* is a system (M, e, m) where M is a set, e is a nullary operation that picks out a distinguished element of M, and m is an associative binary operation that satisfies the identity $m(e, x) = x = m(x, e)$ for every x in M.

A *semiring* is a system $(A, 0, +, 1, \cdot)$ in which $(A, 0, +)$ is a commutative monoid and $(A, 1, \cdot)$ is a monoid satisfying the distributivity of multiplication with respect to addition and 0,

$$
\begin{aligned}
0 \cdot a &= 0 = a \cdot 0 \\
a \cdot (b + c) &= a \cdot b + a \cdot c \\
(a + b) \cdot c &= a \cdot c + b \cdot c
\end{aligned}
$$

[Jan15].

A *ring* is a system $(A, 0, +, -, 1, \cdot)$[4] in which $(A, 0, +, 1, \cdot)$ is a semiring and $(A, 0, +, -)$ is a group [Jan15]. If the operation $\cdot$ is commutative then we refer to the algebra as a *commutative ring*.

A *field* is a ring where every element has a multiplicative inverse, that is, a system $(A, 0, +, -, 1, \cdot, ^{-1})$ in which $(A, 0, +, -)$ and $(A, 1, \cdot, ^{-1})$ are both abelian groups.

For a monoid $(M, 1, \cdot)$, an $(M, 1, \cdot)$-set is a pair (A, α) in which A is a set and $\alpha : M \times A \to A$ a map defined by $\alpha(m, a) = ma$ satisfying $1a = a$ and $m(m'a) = (mm')a$ for all m and m' in M and a in A [Jan15]. This

[4] In some definitions, the multiplicative identity 1 is not included and when it is, we refer to the ring as an *unital* ring.

classical definition of an M-set can be restated to make an M-set an algebraic structure by describing the map α as an M-indexed family of unary operators $(\omega_m)_{m\in M}$ on A where $\omega_m(a) = \alpha(m, a)$ [Jan15].

An M-set can also be defined as a functor. A *left* M-set is an object in the functor category $\mathbf{Sets}^M$ [Jan20]. If $\mathcal{X} : M \to \mathbf{Sets}$ is a covariant functor, then $\mathcal{X}$ specifies an object X in $\mathbf{Sets}$, the image of the unique object of M under $\mathcal{X}$, together with an endomorphism $\mathcal{X}(u) : X \to X$ for each morphism u of M that maps $x \mapsto ux$ such that $\mathcal{X}(u)\mathcal{X}(v) = \mathcal{X}(uv)$ and $\mathcal{X}(e) = 1_X$. There is a similar definition for *right* M-sets as a functor in the presheaf category $\mathbf{Sets}^{M^{op}}$.

Let $(R, 0, +, 1, \cdot)$ be a semiring, then an *R-semimodule* is a system $(A, 0, +, \alpha)$ where $(A, 0, +)$ is a commutative monoid and (A, α) is a $(R, 1, \cdot)$-set satisfying the following identities[5],

$$\begin{aligned} r0 = 0 &= 0a \\ r(a + b) &= ra + rb \\ (r + s)a &= ra + sa \end{aligned}$$

for all $r, s \in R$ and $a, b \in A$ [Jan15].

Let $R = (R, 0, +, -, 1, \cdot)$ be a ring, then an *R-module* is a semimodule over $(R, 0, +, 1, \cdot)$, that is, a system $(A, 0, +, \alpha)$ in which $(A, 0, +)$ is an abelian group and (A, α) is a $(R, 1, \cdot)$-set with $\alpha : R \times A \to A$ a map called *scalar multiplication* that maps $(r, a) \mapsto ra$ such that the identities of a semimodule are satisfied. This definition is specifically that of a *left R-module*. Similarly, we can define a *right R-module*. In general, we will use "R-module" to mean left R-modules. A module is a generalisation of vector spaces where we relax the requirement of the vector space being defined over a field and instead define it over an arbitrary ring[6].

Vector spaces are exactly modules over a field. For a field $K = (K, 0, +, -, 1, \cdot, ^{-1})$, a *$K$-vector space* is a module over $(K, 0, +, -, 1, \cdot)$ [Jan15].

A *semilattice*[7] $(A, 1, \cdot)$ is an idempotent commutative monoid. That is, a

[5] Note that, multiplication of elements in A by elements of the $(R, 1, \cdot)$-set corresponds to "scalar multiplication" (in the case of vector spaces) or, equivalently, the action of the $(R, 1, \cdot)$-set on the monoid $(A, 0, +)$. That is, when writing ra, for $r \in R$ and $a \in A$, we mean $\alpha(r, a)$.

[6] Modules are also a generalisation of abelian groups (as we will see in the section on the category of $\mathbb{Z}$-modules) since abelian groups are exactly modules over the ring of integers.

[7] We sometimes use the notation $(A, 1, \wedge)$ or $(A, 0, \vee)$ and respectively refer to the semilattice as a $\wedge$-semilattice (*meet*-semilattice) or a $\vee$-semilattice (*join*-semilattice) [Jan15].

commutative monoid with $a^2 = aa = a$ for all a in A. A homomorphism of join-semilattices is a map $f : A \to B$ of join-semilattices that preserves finite joins, that is

$$f(x \vee y) = f(x) \vee f(y), \quad f(0) = 0.$$

Similarly, a meet-semilattice homomorphism is a map $f : A \to B$ of meet-semilattices that preserves finite meets, that is

$$f(x \wedge y) = f(x) \wedge f(y), \quad f(1) = 1.$$

Semilattices and semilattice homomorphisms form a category **SemiLat**. We will use the notation for meet-semilattices. We can avoid this notational distinction by referring to the category of semilattices as the category of idempotent commutative monoids and monoid homomorphisms. The category of semilattices is a subcategory of **Pos**, the category of partially ordered sets[8] and monotone maps. The choice of using either meet-semilattices or join-semilattices corresponds to using one of two inclusion functors **SemiLat** $\to$ **Pos**.

4 Free algebras

Consider a vector space V, when X is a basis of V, we say that V is a *free vector space on* X.[9] Now, for any vector space V and W, and a basis X of V, if there is a map $\beta : X \to W$ then, there exists a unique linear map from V to W extending β. That is, there is a unique linear map making the following diagram commute,

$$\begin{tikzcd} V \arrow[r, dashed] & W \\ X \arrow[u, "i"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

where i is the inclusion map $X \to V$. We generalise this example to define a *free algebra* by means of a universal property and will return to the specific case of vector spaces.

[8] A *partially ordered set* (P, R) is a set P together with a relation R which is reflexive, transitive, and antisymmetric (that is, an antisymmetric preorder).

[9] It will be shown that every vector space is free, which makes this specification somewhat redundant.

Let $\mathbb{C}$ be an algebraic category, that is, a full subcategory of $\mathbf{Alg}(\Omega)$. The objects of $\mathbb{C}$ are called $\mathbb{C}$-*algebras*. Let X be a set and $\mathbb{C}[X]$ the category of pairs (A, α), where A is a $\mathbb{C}$-algebra and $\alpha : X \to U(A)$ a map from X to the underlying set of A. A morphism $f : (A, \alpha) \to (B, \beta)$ in $\mathbb{C}[X]$ is an Ω-algebra homomorphism $f : A \to B$ that extends β making the following diagram commute,

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ X \arrow[u, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

that is, such that $f\alpha = \beta$. The *free* $\mathbb{C}$-algebra on X is the initial object in $\mathbb{C}[X]$ [Jan15].

It can be seen from this definition that the property of a $\mathbb{C}$-algebra (A, α) being free means that for every $\mathbb{C}$-algebra (B, β) there exists a unique morphism, extending β, making the following diagram commute

$$\begin{tikzcd} A \arrow[r, dashed] & B \\ X \arrow[u, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

as this is what it means for the $\mathbb{C}$-algebra (A, α) to be the initial object in the category $\mathbb{C}[X]$.

In the case of vector spaces, we used the inclusion map as the map α and, although this is standard in linear algebra, in category theory we generalise this, allowing the use of any map that satisfies the commutative diagram. Such a map will always be injective. We make this adjustment since an initial object is defined up to isomorphism and so we should be able to replace (V, i) with an isomorphic pair, say (V, α). But, now the map α is no longer the inclusion map. We will show that if we change the free $\mathbb{C}$-algebra to an isomorphic one, the new map α will be injective in all but two cases. Consider an initial object (A, α) in $\mathbb{C}[X]$ for which we do not know whether α is injective. We can choose another object (B, β) in which β is injective by taking a $\mathbb{C}$-algebra that is bigger[10] than X and β to be any injective map

[10]The cardinality of $U(B)$ is greater than the cardinality of X.

$X \to B$. This choice of β makes the following diagram commute

$$\begin{tikzcd} A \arrow[r, dashed, "f"] & B \\ X \arrow[u, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

and so, $f\alpha = \beta$. Hence, α must be injective since β is. However, for the empty algebra and the one-element algebra, the map is not injective. In order to make the above construction of β, we need at least one algebra which is bigger than X and if we have an algebra with more than one element, we can take the Cartesian product of it with itself sufficiently many times for it to be bigger than X [Jan22b]. If the algebra has a single element then the Cartesian product of it with itself will also have a single element and so would not be bigger than X.

Consider the category $\mathbb{C}[\emptyset]$ for some category $\mathbb{C}$. The free $\mathbb{C}$-algebra on the empty set is the initial object in $\mathbb{C}$ [Jan15]. More precisely, let A denote the initial object in $\mathbb{C}$, then the free $\mathbb{C}$-algebra on the empty set is the pair $(A, \emptyset_A : \emptyset \to A)$. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ X \arrow[u, "\emptyset_A"] \arrow[ur, "\emptyset_B"'] & \end{tikzcd}$$

where B is any object in $\mathbb{C}$. Since, the empty set is initial in **Sets**, the morphisms $\emptyset_A : \emptyset \to U(A)$ and $\emptyset_B : \emptyset \to U(B)$ will always exist. Hence, if $(A, \emptyset_A)$ is initial in $\mathbb{C}[\emptyset]$, then A is initial in $\mathbb{C}$ since there will always exist a morphism $f : A \to B$ in $\mathbb{C}$ and if A is initial in $\mathbb{C}$ then $(A, \emptyset_A)$ is initial in $\mathbb{C}[\emptyset]$ since, vacuously, the diagram above will always commute.

5 Free semimodules

Let R be a semiring and X a set. In the case of vector spaces, we have a field $\mathbb{K}$ and a set of basis vectors X. We can then form the vector space V by taking all $\mathbb{K}$-linear combinations of the elements of X. We can generalise this procedure and ask what would happen if we took R-linear combinations.

For any given R-linear combination of elements of X, $r_1x_1 + \cdots + r_n + x_n$, we can form a map $u : X \to R$ that maps each x_i to r_i in R. Let $F(X)$ denote the set of all such maps $\{u : X \to R \mid \{x \mid u(x) \neq 0\}$ is finite$\}$ and define a semimodule structure on $F(X)$. Let $\eta_X : X \to F(X)$ be the map that maps each element x of X to the map $\eta_X(x) : X \to R$ which maps x to 1 in R and every other element of X to 0 in R. That is,

$$\eta_X(x) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y. \end{cases}$$

The tuple $(F(X), \eta_X)$ forms an R-semimodule called the *free R-semimodule on X*.

No general method to construct free algebras on a set exists, and so it is difficult for certain algebras. We proceed with a construction of free semimodules as it admits important examples that are studied in mathematics.

Let R be a semiring and $\mathbb{C}$ the category of R-semimodules, the free $\mathbb{C}$-algebra (A, α) on a set X, called the *free R-semimodule* on X, can be constructed as follows [Jan15]. We let A [11] be the set $R^{(X)}$ of all maps $u : X \to R$ such that $\{x \in X : u(x) \neq 0\}$ is finite[12]. We impose this condition as we want this map to be a sum of several elements which come from X and we are not able to define infinite sums[13]. We then define the R-semimodule structure on A by requiring that the operations of the Ω-algebra satisfy the following identities

$$(u + v)(x) = u(x) + v(x),$$
$$(ru)(x) = r(u(x))$$

for every u, v in A, r in R, and x in X. The map $\alpha : X \to A$ is defined by

$$\alpha(x)(y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

where α is a morphism in **Sets**. Now, for any other $\mathbb{C}$-algebra (B, β) in $\mathbb{C}[X]$, there exists a uniquely determined R-semimodule homomorphism $f : R^{(X)} \to B$ that extends β and this homomorphism is given by

$$f(u) = \sum_{x \in X} u(x)\beta(x)$$

[11] Here, A is the set $F(X)$ above.

[12] Equivalently, it is the set of all families $(u_x)_{x \in X}$ of elements in R such that $\{x \in X : u(x) \neq 0\}$ is finite [Jan15].

[13] Defining infinite sums requires limits which require a topology. The structures we are working with do not have a topology defined on them.

[Jan15]. This is the *universal property* of free algebras. Hence, the R-semimodule (A, α) is free, completing our construction. Defining the map α as above, allows us to always construct the R-semimodule homomorphism. To see this, notice that for each map u in A we have that $u = \sum_{x \in X} u(x)\alpha(x)$, since for any y in X,

$$\sum_{x \in X} (u(x)\alpha(x))(y) = \sum_{x \in X} u(x)(\alpha(x)(y))$$

by the R-semimodule structure on A for which every r in R and every map u in A must satisfy the identity $(ru)(x) = r(u(x))$ [14]. Hence,

$$\begin{aligned} \sum_{x \in X} u(x)(\alpha(x)(y)) &= \begin{cases} u(x), & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases} \\ &= u(y) \end{aligned}$$

and so any element u in $R^{(X)}$ can be expressed as a linear combination $u = \sum_{x \in X} u(x)\alpha(x)$. Now, in order to construct the R-semimodule homomorphism f in such a way that it extends β and satisfies $u = \sum_{x \in X} u(x)\alpha(x)$, we have to have that

$$f(u) = f\left(\sum_{x \in X} u(x)\alpha(x)\right) = \sum_{x \in X} u(x) f\alpha(x) = \sum_{x \in X} u(x)\beta(x),$$

and as a result, f is uniquely determined [Jan15].

Given a semiring R, we can form the category $\mathbf{SMod}_R$ of R-semimodules and R-semimodule homomorphisms. A homomorphism of R-semimodules M and N is a map $f : M \to N$ such that

$$f(x + y) = f(x) + f(y) \text{ and } f(rx) = rf(x)$$

for all x and y in M and all r in R.

Consider an R-semimodule in the special case where $R = \{0, 1\}$, then we get a field of characteristic 0. However, if we impose the condition that $1 + 1 = 1$, then we get a 2-element semiring, and so $\{0, 1\}$-semimodules are

[14] Since $\alpha : X \to A$ and $u : X \to R$, $\alpha(x)$ can be identified with an element of A and $u(x)$ can be identified with an element of R.

semilattices [Jan15]. To see this, notice that every $\{0,1\}$-semimodule A is idempotent, since for every $a \in A$,

$$a + a = 1a + 1a = (1+1)a = 1a = a.$$

Conversely, if A is a $\{0,1\}$-semilattice then $1 + 1 = 1$ in A implies that this condition also holds in the semiring $\mathrm{End}(A)$ of endomorphisms[15] of A. This means that sending 0 to 0 and 1 to 1 determines a unique semiring homomorphism $\{0,1\} \to \mathrm{End}(A)$, making A a $\{0,1\}$-semimodule [Jan15]. So, the category of semimodules over $\{0,1\}$ is isomorphic to the category of semilattices.

In this case, the free $\{0,1\}$-semimodule on a set X can be identified with the semilattice $P_{\text{fin}}(X) = (P_{\text{fin}}(X), \emptyset, \cup)$ of finite subsets of X under set union. A map $u : X \to \{0,1\}$ in $\{0,1\}^{(X)}$ corresponds to the finite subset $u^{-1}(1)$ of X, that is the u-preimage of 1, and the addition operation in $\{0,1\}^{(X)}$ corresponds to the set union operation in $P_{\text{fin}}(X)$ [Jan15].

The category of $\mathbb{Z}$-modules is isomorphic to the category $\mathbf{Ab}$ of abelian groups [Jan22a]. There is a forgetful functor $U : \mathbf{Mod}_{\mathbb{Z}} \to \mathbf{Ab}$ that sends a $\mathbb{Z}$-module (A, α) to its underlying abelian group $(A, e, +, i)$ where e and i are determined by $e = \alpha(0, a)$ and $i(a) = \alpha(-1, a)$ and the binary operation $+$ is determined by $a + \cdots + a = \alpha(n, a)$, that is, a added together n-times in the group is the same as $\alpha(n, a)$. This forgetful functor sends a $\mathbb{Z}$-module homomorphism to itself.

Similarly, there is a functor which sends any abelian group to its associated $\mathbb{Z}$-module and any group homomorphism to itself.

The forgetful functor $\mathbf{Mod}_{\mathbb{Z}} \to \mathbf{Ab}$ is a strict inverse of the associated $\mathbb{Z}$-module functor $\mathbf{Ab} \to \mathbf{Mod}_{\mathbb{Z}}$ and so the category of abelian groups and the category of $\mathbb{Z}$-modules are isomorphic.

A free $\mathbb{Z}$-module is isomorphic to the direct sum of n copies of $\mathbb{Z}$, the free abelian group on a single generator, for some natural number n.

6 Vector spaces

We return to our motivating discussion of vector spaces. Let K be a fixed field and V a K-vector space. For any set X, we define the canonical map

[15] A morphism with the same domain and codomain.

$\eta_X : X \to K^{(X)}$ by

$$\eta_X(x)(y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y. \end{cases}$$

This makes the pair $(K^{(X)}, \eta_X)$ into the free K-semimodule on X. That is, for every map $g : X \to V$ there exists a unique linear map $h : K^{(X)} \to V$ (a morphism of K-vector spaces), referred to as the linear map *induced* by g, making the following diagram commute

$$\begin{tikzcd} K^{(X)} \arrow[r, "h"] & V \\ X \arrow[u, "\eta_X"] \arrow[ur, "g"'] & \end{tikzcd}$$

where the induced map h is explicitly defined by

$$h(u) = \sum_{x \in X} u(x)g(x)$$

[Jan15].

A *linearly independent map* is a map $g : X \to V$ such that the induced linear map $h : K^{(X)} \to V$ is injective. A subset S of V is linearly independent if so is the inclusion map $S \to V$ [Jan15]. That is, the map h, making the following diagram

$$\begin{tikzcd} K^{(S)} \arrow[r, "h"] & V \\ S \arrow[u, "\eta_S"] \arrow[ur, "i"'] & \end{tikzcd}$$

commute, is injective. This definition of linear independence in terms of maps can be reconciled with the usual definition by noting that a linear map is injective if and only if it has a trivial kernel. So a map, $g : X \to V$ is linearly independent if and only if for every u in $K^{(X)}$, $h(u) = 0 \Rightarrow u = 0$, that is u is the zero map, and this is true if and only if

$$\sum_{x \in X} u(x)g(x) = 0 \Rightarrow \forall_{x \in X} u(x) = 0.$$

In particular, a subset S of V is linearly independent if and only if for every u in $K^{(S)}$,

$$\sum_{s \in S} u(s)s = \sum_{s \in S} u(s)i(s) = 0 \Rightarrow \forall_{s \in S} u(s) = 0.$$

That is, a subset S of V is linearly independent if and only if we have that, if the linear combination is equal to zero, then all the coefficients $u(s)$ of the linear combination are zero.

A subset S of V is said to be a *basis* of V if it is linearly independent and generates V [Jan15]. Suppose $g : X \to V$ is linearly independent, then $g(X) \subset V$ *generates* V if and only if the induced linear map $h : K^{(X)} \to V$ is surjective [Jan15]. So a subset S of V generates V if and only if the linear map $h : K^{(S)} \to V$ induced by $i : S \to V$ is surjective.

Since a linear map is injective and surjective at the same time if and only if it is an isomorphism, we immediately have two other equivalent characterisations of a basis. A subset S of V is a basis of V if and only if $K^{(S)}$ is isomorphic to V if and only if the pair $(S, i : S \to V)$ is a free K-vector space on S which is exactly the free K-semimodule on S [Jan15]. That is, for any vector space W, there exists a linear map $h : V \to W$ making the diagram

$$\begin{tikzcd}
V \arrow[r, "h"] & W \\
S \arrow[u, "i"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commute[16]. Therefore, to say every K-vector space is free is to say every K-vector space has a basis [Jan15].

A *linear combination* of vectors $v_1, \ldots, v_n$ in V is a K-weighted sum $\sum_i^n k_i v_i$ where each k_i is in K. If S is a basis for V then $K^{(S)}$ is the set of all maps that pick out some combination of elements of K that act as weights for each element s of the basis S. For example, consider the standard n-dimensional Euclidean basis $E = \{e_1, \ldots, e_n\}$, then for $u \in K^{(E)}$ (some choice of K-weights), $u(e_i) = k_i$ for some k_i in K, and a linear combination of the basis vectors in E is a weighted sum $\sum_i^n u(e_i)e_i$.

Now, consider the commutative diagram

$$\begin{tikzcd}
K^{(S)} \arrow[r, "h"] & V \\
S \arrow[u, "\eta_S"] \arrow[ur, "i"'] &
\end{tikzcd}$$

then, $h(u) = \sum_{s \in S} u(s)s$ is a linear combination of vectors s in S. If h is injective then S is linearly independent since each linear combination $h(u)$

[16]We will elaborate further as to why such a linear map exists in the section on *free-forgetful adjunctions*.

corresponds to a unique vector v in V. If h is surjective then S generates V since every vector v in V can be written as a linear combination $h(u)$ for some choice of K-weights u in $K^{(S)}$.

For a set X and a map $g : X \to V$, g picks out basis elements of V if and only if the induced linear map $h : K^{(X)} \to V$ is an isomorphism. That is, g picks out a basis for V if and only if g is linearly independent and $g(X)$ generates V [17].

There are a couple of special cases that should be noted. The empty set and the empty map are always linearly independent [Jan15]. Consider the following commutative diagram

$$\begin{tikzcd}
K^{(\emptyset)} \arrow[r, "h"] & V \\
\emptyset \arrow[u, "\emptyset_{K^{(\emptyset)}}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

and any $u \in K^{(\emptyset)}$, then u is the empty map $\emptyset_K : \emptyset \to K$. Now, the image of $K^{(\emptyset)}$ under the induced map h is the empty set and so h is vacuously injective.

The empty set is a basis for the trivial vector space $\{0\}$ [Jan15]. The empty set is linearly independent from above and the induced linear map $h : K^{(\emptyset)} \to \{0\}$ is surjective since the set $K^{(\emptyset)}$ contains only a single element (the empty map), so the empty set generates $\{0\}$. This coincides with our remark that the free $\mathbb{C}$-algebra on the empty set is the initial object in $\mathbb{C}$ since $\{0\}$ is initial in the category of K-vector spaces $\mathbf{Vect}_K$.

If $X = \{x\}$ is a single element set, then a map $g : X \to V$ is linearly independent if and only if $g(x) \neq 0$, in particular, a single element subset $\{s\}$ of V is linearly independent if and only if $s \neq 0$ [Jan15]. Suppose $g(x) = 0$, then in order for $h\eta_{\{x\}} = g$ to hold, where $\eta_{\{x\}} : X \to K^{(\{x\})}$, we must have that $h(u) = 0$ for every $u \in K^{\{x\}}$ which would imply that the kernel of h is non-trivial, so h is not injective and hence, g is not linearly independent. Now, if g is linearly independent, then h is injective and so it has a trivial kernel, hence x cannot be mapped to 0 in V and so $g(x) \neq 0$.

[17]We can define a basis as a map in this way, however it is rarely done.

We continue with a proof that every K-vector space is free. We first show that every maximal linearly independent subset of a vector space V generates V and using this show that every vector space has a basis and hence, every vector space is free.

Lemma. *If S is a maximal linearly independent subset in a vector space V, then V is generated by S [Jan15].*

Proof. Suppose S is a maximal linearly independent subset of a $\mathbb{K}$-vector space V. We can suppose, without loss of generality, that S is nonempty. If S were empty then for x in V but not in S, there would exist a map $u : \{x\} \to \mathbb{K}$ with $u(x) \neq 0$ since $\{x\}$ is not linearly independent, and so $\{x\}$ is linearly independent for all nonzero x in V, which is a contradiction. So let x be an element of V not in S, then $S \cup \{x\}$ is not linearly independent, since S is the maximal linearly independent subset of V, and so there exists a map $u : S \cup \{x\} \to \mathbb{K}$, such that the set $Y = \{y \in S \cup \{x\} : u(y) \neq 0\}$ is finite and nonempty and $\sum_{y \in Y} u(y)y = 0$. Since S is linearly independent, $u(x) \neq 0$ and the restriction of u on S is a map whose existence contradicts the linear independence of S. We now show that since $u(x)$ is an invertible element of the field $\mathbb{K}$, x is in the subspace of V generated by S. Since $u(x) \neq 0$, $u(x)$ is an invertible element of the field $\mathbb{K}$ and so we can write

$$\begin{aligned} x &= (u(x))^{-1}u(x)x \\ &= (u(x))^{-1}(u(x)x + \sum_{s \in S} u(s)s - \sum_{s \in S} u(s)s) \\ &= (u(x))^{-1}(\sum_{y \in Y} u(y)y - \sum_{s \in S} u(s)s) \\ &= (u(x))^{-1}(\sum_{s \in S} u(s)s) \\ &= \sum_{s \in S} (u(x))^{-1}u(s)s \end{aligned}$$

and so x is in the subspace of V generated by S [Jan15]. $\square$

Hence, a basis of a vector space V is any maximal linearly independent subset of V. We need *Zorn's lemma* to deal with the case of infinite dimensional vector spaces. When the vector space is finitely generated, the proof that every vector space has a basis, can be completed without the use of Zorn's lemma [Jan15].

Lemma (Zorn's Lemma)**.** *Every ordered set, in which every chain is bounded, has a maximal element [Jan15].*

A subset S of an ordered set P is said to be *bounded* if there exists a p in P with $s \leq p$ for all s in S. S is said to be a *chain* if for every s and s' in S, either $s \leq s'$ or $s' \leq s$ in P [18] [Jan15].

Depending on the axioms of set theory that we are using, we are able to place Zorn's lemma as an axiom or deduce it from others. Under certain conditions, Zorn's lemma is equivalent to the *Axiom of Choice* [Jan15].

Theorem. *Every vector space has a basis.*

Proof. Let V be a $\mathbb{K}$-vector space and let $\mathcal{S}$ denote the set of all linearly independent subsets of V. We need to show that every chain in $\mathcal{S}$ is bounded so we can obtain a maximal linearly independent subset that generates V, hence, obtaining a basis for V. If $\underline{S}$ is a chain in $\mathcal{S}$, then $\cup\underline{S}$, its union, is linearly independent (that is $\cup\underline{S} \subset \mathcal{S}$), making $\cup\underline{S}$ the maximal linearly independent subset of V and hence, generates V. To show $\cup\underline{S}$ is linearly independent we have to show that for u in $\mathbb{K}^{(\cup\underline{S})}$ such that $\sum_{s\in\cup\underline{S}} u(s)s = 0$, we have that the finite subset $X = \{x \in \cup\underline{S} : u(x) \neq 0\}$ is empty. That is, there are no $\mathbb{K}$-weights $u(s)$ such that $u(s) \neq 0$ if $\sum_{s\in\cup\underline{S}} u(s)s = 0$. So there is no linear combination of elements in $\cup\underline{S}$ that can produce any other element of $\cup\underline{S}$. Now, to show X is empty, consider that since X is a finite subset of $\cup\underline{S}$ and X is a chain, X is included in some subset S in $\mathcal{S}$. So we have that, $X = \{x \in S : u(x) \neq 0\}$ and $\sum_{s\in S} u(s)s = \sum_{s\in\cup\underline{S}} = 0$. Hence, since S is linearly independent, by restricting u to S, we obtain that X is empty [Jan15]. □

We conclude that every vector space has a basis and so every $\mathbb{K}$-vector space is a free $\mathbb{K}$-semimodule.

7 Fundamental theorem of arithmetic

The natural numbers form a semiring $(\mathbb{N}, 0, +, 1, \cdot)$. Consider the category $\mathbf{SMod}_{\mathbb{N}}$ of semimodules over $\mathbb{N}$. This category is isomorphic to the category of commutative monoids [Jan22a]. Consider any commutative monoid $(M, e, +)$, then it is a semimodule over the semiring of natural numbers where the scalars

[18] That is, S is said to be a *chain* if the order of P induces a linear order on S [Jan15].

are natural numbers and multiplying an element of the monoid by a natural number n is defined for $x \in M$ by

$$nx = x + x + \cdots + x,$$

that is x added to itself n times. So every commutative monoid is uniquely a semimodule over the natural numbers.

Proposition. *The monoid* $(\mathbb{N}, 0, +)$ *is the free commutative monoid on a single element [Jan22a].*

Consider the multiplicative monoid of natural numbers $(\mathbb{N} \setminus \{0\}, 1, \cdot)$, being a commutative monoid, it is a semimodule over the semiring of natural numbers.

Proposition. *The monoid* $(\mathbb{N} \setminus \{0\}, 1, \cdot)$ *is the free commutative monoid on an infinite set, specifically, the set of prime numbers [Jan22a].*

This proposition is equivalent to the *fundamental theorem of arithmetic*: every natural number can be uniquely represented as a product of powers of primes (up to permutation of primes). Let $\mathcal{P}$ denote the set of prime numbers and consider the set $\mathbb{N}^{(\mathcal{P})}$ of maps $u : \mathcal{P} \to \mathbb{N}$ such that only a finite number of primes are mapped by u to a non-zero natural number (the rest of the primes are mapped by u to zero). For each $n \in \mathbb{N}$, by the fundamental theorem of arithmetic we can write

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

for $p_1, \ldots, p_k \in \mathcal{P}$ and $e_1, \ldots, e_k \in \mathbb{N}$. Each map $u \in \mathbb{N}^{(\mathcal{P})}$ maps a prime to a natural number that is its power in the prime decomposition of some natural number. So for each natural number n, there is a map $u \in \mathbb{N}^{(\mathcal{P})}$ such that $u(p_1) = e_1, \ldots, u(p_k) = e_k$ [Jan22c]. For example, the number 30 can be decomposed into primes as $2 \cdot 3 \cdot 5$, so 30 corresponds to the map that sends

$$\begin{aligned} 2 &\mapsto 1, \\ 3 &\mapsto 1, \\ 5 &\mapsto 1, \end{aligned}$$

and every other prime to 0. The number 28 can be decomposed into primes as $4 \cdot 7$, so 28 corresponds to the map that sends

$$\begin{aligned} 2 &\mapsto 2, \\ 3 &\mapsto 0, \\ 5 &\mapsto 0, \\ 7 &\mapsto 1, \end{aligned}$$

and every other prime to 0. The semimodule $\mathbb{N}^{(\mathcal{P})}$ consists of all such maps and it is isomorphic to the semimodule of natural numbers [Jan22c]. This isomorphism expresses the fundamental theorem of arithmetic.

8 Free-forgetful adjunction

Let $\mathbb{A}$ and $\mathbb{B}$ be two categories and $F : \mathbb{A} \to \mathbb{B}$ a functor, for any object B in $\mathbb{B}$, a *universal arrow* $B \to F$ is a pair (A, α) where A is an object in $\mathbb{A}$ and $\alpha : B \to F(A)$ a morphism in $\mathbb{B}$ such that for any object A' in $\mathbb{A}$ and any morphism $\alpha' : B \to F(A')$ in $\mathbb{B}$, there exists a unique morphism $h : A \to A'$ in $\mathbb{A}$ making the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(h)"] & F(A') \\ B \arrow[u, "\alpha"] \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

commute.

Observe that free semimodules are universal arrows. Let X be any set, R a fixed semiring, and $U : \mathbf{SMod}_R \to \mathbf{Sets}$ the forgetful functor, then a *universal arrow* $X \to U$ is a pair (A, α) in which A is a semimodule over R and $\alpha : X \to U(A)$ a map of sets such that for any R-semimodule B and any map of sets $\beta : X \to U(B)$, there exists a unique semimodule homomorphism $h : A \to B$ making the diagram

$$\begin{tikzcd} U(A) \arrow[r, "U(h)"] & U(B) \\ X \arrow[u, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

commute. This corresponds exactly with our definition of the free semimodule on X and so for any set X, the free semimodule on X is the universal arrow $X \to U$ where $U : \mathbf{SMod}_R \to \mathbf{Sets}$ is the forgetful functor.

Adjunctions arise when wanting to compare categories. We begin comparing categories with an isomorphism, requiring that in order for a category $\mathbb{A}$ to be isomorphic to a category $\mathbb{B}$ there must exist an isomorphism $A \approx B$ between them. That is, given a functor $F : \mathbb{A} \to \mathbb{B}$, there exists a functor $G : \mathbb{B} \to \mathbb{A}$ such that $FG = 1_{\mathbb{B}}$ and $GF = 1_{\mathbb{A}}$. When we replace the equalities $FG = 1_{\mathbb{B}}$ and $GF = 1_{\mathbb{A}}$ with natural isomorphisms $FG \approx 1_{\mathbb{B}}$ and $GF \approx 1_{\mathbb{A}}$, then we obtain the notion of *equivalent categories*. We can continue weakening our comparisons by requiring that instead of there being natural isomorphisms $FG \approx 1_{\mathbb{B}}$ and $GF \approx 1_{\mathbb{A}}$, we require that there be natural transformations $FG \to 1_{\mathbb{B}}$ and $1_{\mathbb{A}} \to GF$, and then the categories are no longer equivalent. However, this mode of comparison is still of interest and comes up in a number of seemingly disconnected areas in mathematics. This is called an *adjunction* and F and G are said to be *adjoint functors*.

There are a number of equivalent definitions of an adjunction between two categories. The ones we will be using are the following.

Definition.

1. *For categories $\mathbb{X}$ and $\mathbb{A}$, an adjunction $\mathbb{A} \to \mathbb{X}$ is a triple (F, U, η), in which $F : \mathbb{X} \to \mathbb{A}$ and $U : \mathbb{A} \to \mathbb{X}$ are functors, and $\eta : 1_{\mathbb{X}} \to UF$ a natural transformation such that each $(F(X), \eta_X)$ is a natural transformation [Jan20].*

2. *An adjunction between categories $\mathbb{X}$ and $\mathbb{A}$ is a four-tuple $(F, U, \eta, \varepsilon)$ where $F : \mathbb{X} \to \mathbb{A}$ and $U : \mathbb{A} \to \mathbb{X}$ are functors and $\eta : 1_{\mathbb{X}} \to UF$ and $\varepsilon : FU \to 1_{\mathbb{A}}$ are natural transformations called the unit and counit, respectively, such that the following two diagrams*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FUF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} UFU \arrow[d, "U\varepsilon"'] & U \arrow[l, "\eta U"'] \arrow[dl, "1_U"] \\ U & \end{tikzcd}$$

   commute [Jan20].

3. *An adjunction between categories $\mathbb{X}$ and $\mathbb{A}$ is a functor $F : \mathbb{X} \to \mathbb{A}$ and a family $((U(A), \psi_A))_{A \in \mathbb{A}_0}$, in which each $U(A)$ is an object in $\mathbb{X}$ and each ψ_A is an isomorphism $\hom_{\mathbb{X}}(-, U(A)) \to \hom_{\mathbb{A}}(F(-), A)$ [Jan20].*

The functors F and U are called the *left* and *right adjoint* (of U and of F), respectively, and the natural transformations η and ε are called the *unit* and *counit*, respectively.

The third definition can be derived from the definition of an adjunction between categories $\mathbb{X}$ and $\mathbb{A}$ as a triple (F, U, φ), in which $F : \mathbb{X} \to \mathbb{A}$ and $U : \mathbb{A} \to \mathbb{X}$ are functors, and $\varphi : \hom_{\mathbb{A}}(F(*), -) \to \hom_{\mathbb{X}}(*, U(-))$ a natural isomorphism. The functors $\hom_{\mathbb{A}}(F(*), -)$ and $\hom_{\mathbb{X}}(*, U(-))$ are the composites

$$\mathbb{X}^{op} \times \mathbb{A} \xrightarrow{F^{op} \times \mathbb{X}} \mathbb{A}^{op} \times \mathbb{A} \xrightarrow{\hom} \mathbf{Sets}$$

and

$$\mathbb{X}^{op} \times \mathbb{A} \xrightarrow{\mathbb{X} \times U} \mathbb{X}^{op} \times \mathbb{X} \xrightarrow{\hom} \mathbf{Sets},$$

respectively [Jan20]. The definition dual to this is a triple (F, U, ψ) where F and U are as above, and $\psi : \hom_{\mathbb{X}}(*, U(-)) \to \hom_{\mathbb{A}}(F(*), -)$ is a natural isomorphism. The most obvious way to define ψ is to take it as the inverse of φ.

The following theorem enables us to define an adjunction between categories $\mathbb{A}$ and $\mathbb{X}$ if we have a functor $U : \mathbb{A} \to \mathbb{X}$ and a universal arrow $X \to U$ for each object X in $\mathbb{X}$.

Theorem. *Let $\mathbb{A}$ and $\mathbb{X}$ be categories. Given a functor $U : \mathbb{A} \to \mathbb{X}$ and a family $((F(X), \eta_X))_{X \in \mathbb{X}_0}$ in which each $(F(X), \eta_X)$ is a universal arrow $X \to U$, there exists a unique way to define F also on morphisms such that it becomes a functor and the family $(\eta_X)_{X \in \mathbb{X}_0}$ becomes a natural transformation $\eta : 1_{\mathbb{X}} \to UF$ [Jan20].*

Let R be a fixed semiring and $U : \mathbf{SMod}_R \to \mathbf{Sets}$ the forgetful functor. We are able to use the above theorem to define an adjunction between U and F by defining F in such a way that it is the right adjunction of U.

Consider that for any set X, the universal arrow $X \to U$ is a pair (A_X, η_X). So there is a family of universal arrows $((A_X, \eta_X))_{X \in \mathbf{Sets}_0}$. We can define a map $F : \mathbf{Sets} \to \mathbf{SMod}_R$ on the objects of $\mathbf{Sets}$ that maps any set X to the set A_X, where A_X comes from the universal arrow $X \to U$. Then the family of universal arrows can be rewritten $((F(X), \eta_X))_{X \in \mathbf{Sets}_0}$ and so by the above theorem, there exists a unique way to define F also on morphisms so that it becomes a functor and the family $(\eta_X)_{X \in \mathbf{Sets}_0}$ becomes a natural transformation $\eta : 1_{\mathbf{Sets}} \to UF$. Hence, making the triple (F, U, η) an adjunction $\mathbf{Sets} \to \mathbf{SMod}_R$.

We define F on a morphism $f : X \to Y$ in **Sets** in such a way that the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UF(X) \arrow[d, "UF(x)"] \\ Y \arrow[r, "\eta_Y"'] & UF(Y) \end{tikzcd}$$

commutes. This uniquely defines F on the morphisms of **Sets** in such a way that η is a natural transformation [Jan20]. So, for any map of sets $f : X \to Y$, the linear map $F(f) : F(X) \to F(Y)$ is defined by

$$F(f)\left(\sum_{x_i \in X} r_i x_i\right) = \sum_{x_i \in X} r_i f(x_i).$$

In the case of vector spaces, this means that any map of bases induces a linear map between the vector spaces generated by those bases.

The *unit* natural transformation $\eta : 1_{\mathbb{X}} \to UF$ of any adjunction $(F, U, \eta, \varepsilon)$ satisfies the following universal property. If X is any object in $\mathbb{X}$, and A is any object in $\mathbb{A}$, and $\alpha : X \to U(A)$ is any morphism in $\mathbb{X}$, then there exists a unique morphism $f : F(X) \to A$ in $\mathbb{A}$ making the diagram

$$\begin{tikzcd} UF(X) \arrow[r, "U(f)"] & U(A) \\ X \arrow[u, "\eta_X"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commute. That is, for every object X in $\mathbb{X}$, the pair $(F(X), \eta_X)$ is a universal arrow $X \to U$.

The forgetful functor $U : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Sets}$, for $\mathbb{K}$ a fixed field, has a left adjoint $F : \mathbf{Sets} \to \mathbf{Vect}_{\mathbb{K}}$ that maps any set X to $F(X)$, the free vector space[19] on X and any map of sets $f : X \to A$ to the linear map $F(f) : F(X) \to F(A)$ with

$$F(f)\left(\sum_{x_i \in X} r_i x_i\right) = \sum_{x_i \in X} r_i f(x_i).$$

[19] Given a set X, we can form a vector space on X consisting of all formal $\mathbb{K}$-linear combinations of elements of X with a vector space structure on it as done in the earlier section on vector spaces.

It can be checked that F is indeed a functor and is appropriately called the *free functor*. This adjunction is known as the *free-forgetful adjunction*.

In order for F to be the left adjoint to the forgetful functor U, it needs to be the case that, for every $\mathbb{K}$-vector space V and every set X, the following bijections

$$\psi_V(X) : \hom_{\mathbf{Sets}}(X, U(V)) \approx \hom_{\mathbf{Vect}_{\mathbb{K}}}(F(X), V)$$

hold. So, for every map of sets $f : X \to U(V)$, we need a linear map $\psi_V(X)(f) : F(X) \to V$. We define this map as

$$\psi_V(X)(f)\left(\sum_{x_i \in X} r_i x_i\right) = \sum_{x_i \in X} r_i f(x_i).$$

Now, conversely, given a linear map $g : F(X) \to V$ we need a map of sets $\varphi_V(X)(g) : X \to U(V)$ so that the map $\varphi_V(X) : \hom_{\mathbf{Vect}_{\mathbb{K}}}(F(X), V) \to \hom_{\mathbf{Sets}}(X, U(V))$ is inverse to $\psi_V(X)$ and this isomorphism is natural. This map is given by $\varphi_V(X)(g) = g$.

The unit $\eta : 1_{\mathbf{Sets}} \to UF$ of the free-forgetful adjunction $(F, U, \eta, \varepsilon) : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Sets}$ is a natural transformation consisting of component functions $\eta_X : X \to UF(X)$ for each set X that are each the inclusion of X into the underling set of $F(X)$, the free vector space on X.

The *counit* $\varepsilon : FU \to 1_{\mathbf{Vect}_{\mathbb{K}}}$ of the free-forgetful adjunction $(F, U, \eta, \varepsilon) : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Sets}$ is a natural transformation consisting of component linear transformations $\varepsilon_V : FU(V) \to V$ for each vector space V. Since $FU(V)$ is the vector space with one basis vector for each element in $U(V)$, the linear map ε_V takes linear combinations of the elements in the set $U(V)$ and views them as vectors in V. Each linear map ϵ_V satisfies the following universal property. If V is any $\mathbb{K}$-vector space and X is any set, and $f : F(X) \to V$ is any linear transformation, then there is a unique map of sets $h : X \to U(V)$ making the diagram

$$\begin{tikzcd}
FU(V) \arrow[d, "\varepsilon_V"'] & F(X) \arrow[l, "F(h)"'] \arrow[dl, "f"] \\
V &
\end{tikzcd}$$

commute. That is, the pair $(U(V), \varepsilon_V)$ is a universal arrow $F \to V$ for every

$\mathbb{K}$-vector space V [20]. We can interpret the counit as imposing the condition that linear combinations of vectors in V themselves be vectors in V.

[20] This explains the nomenclature "unit" and "counit" as the two notions are dual.

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